

Vacuum static compactified wormholes in eight-dimensional Lovelock theory

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Abstract

In this paper new exact solutions in eight dimensional Lovelock theory will be presented. These solutions are vacuum static wormhole, black hole and generalized Bertotti-Robinson space-times with nontrivial torsion. All the solutions have a cross product structure of the type $M_5 \times \Sigma_3$ where M_5 is a five dimensional manifold and Σ_3 a compact constant curvature manifold. The wormhole is the first example of a smooth vacuum static Lovelock wormhole which is neither Chern-Simons nor Born-Infeld. It will be also discussed how the presence of torsion affects the "navigableness" of the wormhole for scalar and spinning particles. It will be shown that the wormhole with torsion may act as "geometrical filter": a very large torsion may "increase the traversability" for scalars while acting as a "polarizator" on spinning particles. This may have interesting phenomenological consequences.

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1 Introduction

Since the ideas of Kaluza and Klein and with the advent of string theory, the possibility to have extra-dimensions became one of the most promising possibility to extend the standard model of particles physics. Higher dimensional theories of gravity may present some new features which are absent in four dimensions. Indeed in four dimensions the only gravitational action that can be built from curvature invariants leading to second order equations in the metric is the Einstein-Hilbert action. The situation changes in higher dimensions. In five dimensions one can add for example a Gauss-Bonnet term to the action which is quadratic in the curvature and leads to second order equations. In higher dimensions one can add higher curvature powers to the action. Such higher curvature power theories leading to second order equations for the metric are known as Lovelock theories [1] (see, for pedagogical reviews on Lovelock gravities, [2] [3] [4]).

Unlike General Relativity, the Lovelock equations of motion do not imply the vanishing of torsion. However, the Lovelock equations of motions put very strong constraints on the torsion so that it is very difficult to find exact solutions with torsion. The only case where such consistency conditions are automatically satisfied is when in odd dimensions the coupling constants are tuned in such a way that the theory becomes a Chern-Simons theory (see, for instance, [3] [4]). In this case the theory possesses only one maximally symmetric vacuum as well as an enhanced gauge symmetry. Because of the degeneracy of the theory it is easier to find vacuum solutions with nontrivial torsion (however, often the torsion turns out to be pure gauge). Following a suggestion in [5] based on an analogy with BPS states in Yang-Mills theory, the first vacuum solution with torsion in a non-Chern-Simons theory was found in [6]. This solution is the purely gravitational analogue of the Bertotti-Robinson space-time. The crucial point of the construction was to make the ansatz of a torsion concentrated on a three-dimensional sub-manifold (according to the "BPS prescription" of [5]) so that many of the torsion equations are identically satisfied. It is therefore interesting to see if also more general solutions than the one mentioned can be found in the non-Chern-Simons case as e.g. black holes or wormholes. Indeed a black hole solution with the "BPS torsion" in five dimension turns out to be Chern-Simons [7].

An interesting possibility worth to be further analyzed is to try to generalize the construction of [5], [6] and [7] in higher dimensions. The even-

dimensional case is particularly interesting since, in these cases, one avoids in part the degeneracies of Chern-Simons theory as there is no enhanced gauge symmetry in this case. Thus, we will consider the eight-dimensional case: one can add another 3-D compact sub-manifolds which implies that there are now two possibilities to put "BPS torsion". We will search for solutions in eight dimensions with the structure of $M_5 \times \Sigma_3$ where $M_5 \equiv M_2 \times F(r)N_3$ is a five dimensional manifold, where M_2 plays the role of an $r - t$ plane, N_3 is a constant curvature manifold (which we will call base manifold) and $F(r)$ is a warp factor. Σ_3 is another compact constant curvature manifold that plays the role of a compactified 3-D space. Of course, such eight dimensional solutions are non-Chern-Simons as in even dimensions Chern-Simons theories cannot exist. In even dimensions Born-Infeld gravity is the most similar theory to Chern-Simons gravity since this theory admits only one maximally symmetric vacuum (see, for instance, [3] [4]). It will therefore be of special interest to find exact solutions with nontrivial torsion in the non-Born-Infeld case.

Lovelock theories admit static vacuum wormhole solutions (static wormholes in gravity in dimension higher than four were found in [8], [9] in the case of Chern-Simons theory. Indeed one can see that for the Einstein-Gauss-Bonnet theory vacuum wormholes exist in any dimension provided that there is a unique maximally symmetric vacuum [10]). Wormhole solutions are interesting in that they heavily try out the geometric structure of the theory. Indeed in four dimensions, static wormhole solutions cannot exist in vacuum and the matter that sustains such solutions violates the energy conditions (for a nice review see [11]; in Einstein-Gauss-Bonnet theory there exist non-vacuum wormhole solutions whose energy-momentum tensors respect the energy conditions [12]). It is therefore interesting to search for smooth static vacuum wormhole solutions in more general Lovelock theories which have a quite rich dynamical content. We will construct in this paper eight dimensional exact static vacuum solutions which have the structure of $M_5 \times \Sigma_3$. Some of these exact static vacuum solutions can be considered as effective five dimensional wormholes with three compactified extra dimensions playing a "spectator" role. We will show that these solutions can carry non-trivial torsion. A remarkable feature of the present construction is that the presence of torsion affects the "navigableness" of the wormhole: torsion has quite different effects on scalars and spinning particles. A large torsion improves the "navigableness" for scalars while acting on spinning particles, in a sense, as a "polarizator". In very much the same way as the coupling of a magnetic

field with a Fermion, the coupling of torsion with angular momenta favours angular momenta "pointing in the same direction as" torsion: a throat with torsion can act as a filter for particles which have not the right polarization (with respect to the background torsion).

The same geometric structure of $M_5 \times \Sigma_3$ can also support effective five-dimensional black hole solutions which can have non-trivial torsion. Another simple class of solutions with torsion has the structure of $M_2 \times S_3 \times S_3$ where M_2 is a two-dimensional constant curvature Lorentzian manifold and can be seen as a generalized Bertotti-Robinson space-time.

The structure of the paper will be the following: In the second section we will give a short review of eight dimensional Lovelock theory. In third and fourth section the curvature and the torsion and the corresponding equations of motion for a manifold of the form $M_5 \times \Sigma_3$ will be discussed. In the fifth and sixth sections the wormhole solutions and their "navigableness" will be analyzed. Then the black hole solutions and the generalized Bertotti-Robinson solutions will be shortly described. Eventually, some conclusive remarks will be given.

2 Eight dimensional Lovelock theory

The most general Lovelock action in 8-D in reads

$$I = \int \epsilon_{ABCDEFGH} \left(\frac{c_0}{8} e^A e^B e^C e^D e^E e^F e^G e^H + \frac{c_1}{6} R^{AB} e^C e^D e^E e^F e^G e^H + \frac{c_2}{4} R^{AB} R^{CD} e^E e^F e^G e^H + \frac{c_3}{2} R^{AB} R^{CD} R^{EF} e^G e^H \right) \quad (1)$$

where in the second order formalism the quadratic (Gauss-Bonnet) term proportional to c_2 reads

$$R^{AB} R^{CD} e^E e^F e^G e^H \epsilon_{ABCDEFGH} = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} \quad (2)$$

and the cubic term proportional to c_3 reads

$$\begin{aligned} R^{AB} R^{CD} R^{EF} e^G e^H \epsilon_{ABCDEFGH} = & R^3 + 3RR^{\mu\nu\alpha\beta} R_{\alpha\beta\mu\nu} - 12RR^{\mu\nu} R_{\mu\nu} \\ & + 24R^{\mu\nu\alpha\beta} R_{\alpha\mu} R_{\beta\nu} + 16R^{\mu\nu} R_{\nu\alpha} R_{\mu}^{\alpha} + 24R^{\mu\nu\alpha\beta} R_{\alpha\beta\nu\rho} R_{\mu}^{\rho} \\ & + 8R^{\mu\nu}_{\alpha\rho} R^{\alpha\beta}_{\nu\sigma} R^{\rho\sigma}_{\mu\beta} + 2R_{\alpha\beta\rho\sigma} R^{\mu\nu\alpha\beta} R^{\rho\sigma}_{\mu\nu} \end{aligned} \quad (3)$$

where $R^{\alpha\beta\mu\nu}$, $R^{\mu\nu}$ and R are respectively the Riemann tensor the Ricci tensor and the Ricci scalar in the second order formalism.

The equations of motion varying the action with respect to the vielbein are

$$\epsilon_H = \epsilon_{ABCDEFGH} (c_0 e^A e^B e^C e^D e^E e^F e^G + c_1 R^{AB} e^C e^D e^E e^F e^G + c_2 R^{AB} R^{CD} e^E e^F e^G + c_3 R^{AB} R^{CD} R^{EF} e^G) = 0 \quad (4)$$

And varying with respect to the spin connection one obtains

$$\epsilon_{GH} = \epsilon_{ABCDEFGH} T^A (c_1 e^B e^C e^D e^E e^F + 2c_2 R^{BC} e^D e^E e^F + 3c_3 R^{BC} R^{DE} e^F) = 0. \quad (5)$$

where the torsion two form $T^A = de^A + \omega^A_B e^B$ is related to the antisymmetric part of the Christoffel symbols by

$$T^\lambda_{\mu\nu} \equiv e_A^\lambda T^A_{\mu\nu} = 2\Gamma^\lambda_{[\mu\nu]} \quad (6)$$

In even dimension a Lovelock theory is called Born-Infeld theory if the coupling constants are tuned in such a way that the theory admits only one maximally symmetric vacuum. In eight dimensions the Born-Infeld Lagrangian has the form

$$I_{BI} = \int \epsilon_{ABCDEFGH} (R^{AB} + \Lambda e^A e^B) \cdot (R^{CD} + \Lambda e^C e^D) (R^{EF} + \Lambda e^E e^F) (R^{GH} + \Lambda e^G e^H).$$

If we compare this action with (1) it is easy to find the tuning of the coupling constants leading to Born-Infeld action

$$c_2^2 = 3c_1 c_3; \quad c_1^2 = 3c_0 c_2. \quad (7)$$

The Born-Infeld Lagrangians in even dimensions and the Chern-Simons Lagrangians in odds dimensions share some interesting features (see, for instance, [3], [4]). First of all, in both cases there is a unique maximally symmetric vacuum. One can also argue that both types of Lagrangians have some degree of degeneracy in the sense that it may happen that the equations of motion leave undetermined some of the metric functions. Thus, one may expect that to construct static vacuum wormholes in the Born-Infeld and Chern-Simons cases can be quite easier than in the generic case (up to

now, the only static vacuum wormhole is the one in [8], [9], [10] in which a certain degree of degeneracy is manifest). In the generic case the wormhole structure imposes very strong constraints on the metric functions and on the energy momentum tensor (see, for instance, [11]). In five dimensions, only recently it appeared the first stationary Ricci flat wormhole [13]. For these reasons, we will construct here static vacuum wormhole spacetimes in non-Born-infeld cases (namely, when the conditions in Eq. (7) are not fulfilled). We will also consider the effects of torsion which, in higher dimensions, is generically different from zero. In the generic non-Chern-Simons Lovelock case, the equations of motions for the torsion are very restrictive and until very recently, no exact solution with torsion was known. The first was discovered in [6] using the ansatz for the torsion (inspired by an analogy with gauge theory first proposed in [5])

$$T^i = K(r)\epsilon^{ijk}e_je_k. \quad (8)$$

3 Ansatz of $M_5 \times S_3$ manifold

We make the ansatz for the metric

$$ds^2 = -(f(r))^2 dt^2 + \frac{dr^2}{(g(r))^2} + r^2 d\Sigma_1^2 + d\Sigma_2^2$$

where Σ_1 (which, from now on, we will call "*base manifold*") and Σ_2 are two constant curvature 3-D manifolds. We can choose the vielbein as (using the indices i, j, k for Σ_1 and a, b, c for Σ_2)

$$e^0 = f dt, \quad e^1 = \frac{dr}{g}, \quad e^i = r \hat{e}^i, \quad e^a = \hat{e}^a \quad (9)$$

so that the Riemannian part of the connection is

$$\omega^0_1 = g \frac{(f)'}{f} e^0, \quad \omega^i_1 = \frac{g}{r} e^i, \quad \omega^i_j = \hat{\omega}^i_j, \quad \omega^a_b = \hat{\omega}^a_b, \quad (10)$$

where we used the notation

$$f' = \partial_r f, \quad g' = \partial_r g,$$

and

$$\omega^a_{1} = 0 = \omega^a_{0} = \omega^i_{0} = 0 = \omega^i_{a}$$

where \hat{e}^i and \hat{e}^a are the intrinsic vielbeins respectively of the base manifold Σ_1 and of Σ_2 and $\hat{\omega}^i{}_j$ and $\hat{\omega}^a{}_b$ are the intrinsic spin connections of the base manifold Σ_1 and of Σ_2 . For the torsion we make the following ansatz

$$T^0 = T^1 = 0, \quad T^i = K_1(r)\epsilon^{ijk}e_j e_k, \quad T^a = K_2(r)\epsilon^{abc}e_b e_c \quad (11)$$

so that the contorsion is

$$K^{ij} = -K_1(r)\epsilon^{ijk}e_k, \quad K^{ab} = -K_2(r)\epsilon^{abc}e_c, \quad (12)$$

and the total connection is

$$(\omega_{tot})^{AB} = \omega^{AB} + K^{AB}. \quad (13)$$

With this ansatz the total curvature two forms (which includes torsion effects) take the form

$$\begin{aligned} R^{0a} &= R^{1a} = R^{ia} = 0 \\ R^{01} &= -\frac{g}{f}(gf')'e^0 e^1 \\ R^{0i} &= -\frac{g^2 f'}{fr}e^0 e^i \\ R^{1i} &= -\frac{gg'}{r}e^0 e^i - \frac{1}{r}T^i \\ R^{ij} &= \left(\frac{\gamma - g^2}{r^2}\right)e^i e^j - \frac{d(rK_1)}{r}\epsilon^{ijk}e_k - K_1^2 e^i e^j \\ R^{ab} &= \eta e^a e^b - d(K_2)\epsilon^{abc}e_c - K_2^2 e^a e^b \end{aligned}$$

where γ and η are the intrinsic scalar curvatures of Σ_1 and Σ_2 respectively.

It is worth to stress here that torsion can be divided into its irreducible components: it is trivial to see that when torsion has the form in Eq. (11) it is fully skew-symmetric so that both its trace and its symmetric parts vanish: in these cases (see for instance [14]) one says that only the axial part of the torsion is present.

4 Equations of motion

It is easy to see that in the equations of motions Eq. (4), in all the terms in which it appears R^{1i} the non-Riemannian part proportional to T^i do not contribute to the equations of motion due to the identities satisfied by ansatz used for the torsion (see [5]). In order to satisfy the equations of motion one must have

$$d(rK_1) = 0; \quad dK_2 = 0 \Rightarrow \quad (14)$$

$$T^i = \frac{\delta_{(1)}}{r} \epsilon^{ijk} e_j e_k, \quad (15)$$

$$T^a = K_2 \epsilon^{abc} e_b e_c \quad (16)$$

where $\delta_{(1)}$ and K_2 are constants.

It is worth to stress here the following point: the constants γ and η can be rescaled to -1 , 0 and 1 (according to the scalar curvatures of Σ_1 and Σ_2). However, the "torsion" constants $\delta_{(1)}$ and K_2 cannot be rescaled away, they can take any real values (compatible with the equations of motion) since they are true integration constants representing the strength of the torsion in the i and the a directions. Thus, as in [6], the presence of torsion will be manifest directly in the metric: this is unlike the five-dimensional Chern-Simons case in which in the half BPS black hole constructed in [7], torsion manifests itself mainly in the Killing spinor equation.

It is convenient to introduce the following definitions in order to simplify the notation

$$R^{01} \equiv F e^0 e^1; \quad R^{0i} \equiv A e^0 e^i; \quad R^{1i} \equiv B e^1 e^i - \frac{1}{r} T^i \quad (17)$$

$$R^{ij} \equiv D(r) e^i e^j; \quad R^{ab} \equiv \tilde{\eta} e^a e^b \Rightarrow \quad (18)$$

$$A = -\frac{g^2 f'}{f r}, \quad B = -\frac{g g'}{r}, \quad (19)$$

$$F = -\frac{g}{f} (g f')', \quad D = \frac{\tilde{\gamma} - g^2}{r^2}, \quad (20)$$

where $\tilde{\gamma} = \gamma - (\delta_{(1)})^2$ and $\tilde{\eta} = \eta - K_2^2$ are the effective curvatures (shifted by the fluxes of torsion). We can now write the equations of motion for our ansatz:

$$\epsilon_0 = B [20c_1 + 4c_2 (D + 3\tilde{\eta}) + 12c_3 \tilde{\eta} D]$$

$$+ [140c_0 + 20c_1D + 20c_1\tilde{\eta} + 12c_2\tilde{\eta}D] = 0, \quad (21)$$

$$\begin{aligned} \epsilon_1 &= A [20c_1 + 4c_2 (D + 3\tilde{\eta}) + 12c_3\tilde{\eta}D] \\ &+ [140c_0 + 20c_1D + 20c_1\tilde{\eta} + 12c_2\tilde{\eta}D] = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} \epsilon_i &= F (20c_1 + 4c_2D + 12c_2\tilde{\eta} + 12c_3\tilde{\eta}D) + A (40c_1 + 24c_2\tilde{\eta}) \\ &+ AB (8c_2 + 24c_3\tilde{\eta}) + B (40c_1 + 24c_2\tilde{\eta}) \\ &+ (420c_0 + 20c_1D + 60c_1\tilde{\eta} + 12c_2\tilde{\eta}D) = 0, \end{aligned} \quad (23)$$

$$\epsilon_a = F (20c_1 + 12c_2D + 4c_2\tilde{\eta} + 12c_3\tilde{\eta}D)$$

$$\begin{aligned} &+ A (60c_1 + 12c_2\tilde{\eta} + 12c_2D + 12c_3\tilde{\eta}D) \\ &+ AB (24c_2 + 24c_3\tilde{\eta}) + B (60c_1 + 12c_2\tilde{\eta} + 12c_2D + 12c_3\tilde{\eta}D) \\ &+ (420c_0 + 60c_1D + 20c_1\tilde{\eta} + 12c_2\tilde{\eta}D) = 0. \end{aligned} \quad (24)$$

As far as the equations in which torsion appears explicitly is concerned, a very nice feature of the ansatz for the torsion (8) is that almost all the equations (5) are identically fulfilled. The only components of the torsion equations (5) which do not vanish identically because of the ansatz of the torsion read

$$\epsilon_{ij} = 0 \Rightarrow \delta_{(1)} (F (4c_2 + 12c_3\tilde{\eta}) + 20c_1 + 12c_2\tilde{\eta}) = 0, \quad (25)$$

$$\begin{aligned} \epsilon_{ab} = 0 \Rightarrow & K_2 (F (4c_2 + 12c_3D) + B (12c_2 + 12c_3D) + A (12c_2 + 12c_3D) \\ & + 24c_3AB + 20c_1 + 12c_2D) = 0 \end{aligned} \quad (26)$$

It is important to notice that Eqs. (21) and (22) are equal provided B is exchanged with A . Therefore in order to be compatible there are two possibilities:

the first is $A = B$ which implies $f = g$ and so it corresponds to black hole like solutions.

The second possibility appears when in the two square brackets in Eqs. (21) and (22) are zero separately leaving open the possibility to have worm-hole solutions.

We will discuss the two possibilities separately.

5 Wormholes

In order to have $f \neq g$ the two square brackets in Eqs. (21) and (22) must be zero separately: the reason is that in the generic case the consistency of Eqs. (21) and (22) imply $A = B$ (where A and B are defined in Eqs. (17), (19) and (20)) and then $f = g$. Thus, in order to avoid this "no-go argument" for the appearance of wormhole, one has to ask that

$$(4c_2 + 12c_3\tilde{\eta}) D(r) = -(20c_1 + 12c_2\tilde{\eta}), \quad (27)$$

$$(20c_1 + 12c_2\tilde{\eta}) D(r) = -(140c_0 + 20c_1\tilde{\eta}). \quad (28)$$

The consistency condition of Eqs. (27) and (28) is

$$\frac{20c_1 + 12c_2\tilde{\eta}}{4c_2 + 12c_3\tilde{\eta}} = \frac{140c_0 + 20c_1\tilde{\eta}}{20c_1 + 12c_2\tilde{\eta}} \quad (29)$$

which is nothing but the requirement that Eqs. (21) and (22) should be identically satisfied no matter the values of A and B . This condition fixes the function $D(r)$ to be a constant D_0 as follows

$$D(r) \equiv D_0 = -\frac{20c_1 + 12c_2\tilde{\eta}}{4c_2 + 12c_3\tilde{\eta}} = -\frac{140c_0 + 20c_1\tilde{\eta}}{20c_1 + 12c_2\tilde{\eta}} \quad (30)$$

Using the definition of $D(r)$ in Eqs. (18) and (20), we obtain the form of g^2

$$g^2 = -D_0 r^2 + \tilde{\gamma} \quad (31)$$

In the case of vanishing torsion (in which $\tilde{\gamma} = \gamma$ and $\tilde{\eta} = \eta$), in order for the metric to represent a static wormhole, a necessary condition is that the g_{rr} metric component has the form in Eq. (31) with both D_0 and γ negative (see, for instance, [11], [8]).

There are two possibilities:

the first possibility is to satisfy the constraint (29) so that the g_{rr} component is determined by Eq. (31).

The second possibility is when the round brackets in Eqs. (27) and (28) vanish identically leaving D indeterminate. This happens for the following tuning of the couplings and $\tilde{\eta}$

$$5c_1 + 3c_2\tilde{\eta} = 0; \quad c_2 + 3c_3\tilde{\eta} = 0; \quad 7c_0 + c_1\tilde{\eta} = 0 \quad (32)$$

which imply that

$$\tilde{\eta} = -\frac{c_2}{3c_3}$$

together with two relations involving only the couplings

$$c_2^2 = 5c_1c_3; \quad 5c_1^2 = 21c_2c_0 \quad (33)$$

At a first glance, one would expect that such a strong degeneracy condition should correspond to the Born-Infeld case which is the even-dimensional analogous of the Chern-Simons Lagrangians (see, for instance, [3] [4]). Often, this implies that the field equations manifest a huge degeneracy in such a way that the metric (representing a given exact solution) may have arbitrary functions left completely undetermined by the field equations themselves. As a matter of fact, the first exact vacuum solution representing a static wormhole has been found in the Chern-Simons case [8]. Thus, when searching vacuum static wormholes, one may think that the only hope to find them is in the cases of Born-Infeld or Chern-Simons. Remarkably enough, the relations above *do not correspond to the Born-Infeld tunings* (7). We will treat the degenerate and non-degenerate case separately

5.1 Degenerate wormhole

In this and in the following subsections we will consider wormholes characterized by the fact that

$$f = \text{const}, \quad D(r) = \text{const} \equiv D_0.$$

where $D(r)$ is defined in Eq. (18). The reason is that these are the simplest and most elegant wormholes in which one can see in the clearest possible way the different effects of torsion on the "throat navigableness" of scalar and spinning particles. As it will be explained in the next sub-section, in the generic case f satisfies a hypergeometric-like equation. Thus, wormhole solutions correspond to hypergeometric functions without zeros in r . However, from the qualitative point of view, wormhole solutions with a non-constant f do not have new features if compared to wormhole solutions with f constant.

The easiest way to proceed is to first solve the torsion equations: the ij component of the torsion equations (namely, Eq. (25)) is identically satisfied because of the degeneracy condition in Eq. (33). The ab component of the torsion equations (that is, Eq. (26)) is

$$\epsilon_{ab} = 12c_3D_0^2 + 24c_2D_0 + 20c_1 = 0, \quad (34)$$

This fixes the constant D_0 in terms of the coupling constants c_i . When the degeneracy conditions in Eq. (33) hold, Eqs. (21) and (22) are identically satisfied; it also is immediate to see that, due to the degeneracy conditions in Eq. (33), Eq. (23) is identically satisfied as well. Only Eq. (24) is left:

$$D_0^2(12c_2 + 12c_3\tilde{\eta}) + D_0(120c_1 + 24c_2\tilde{\eta}) + (420c_0 + 20c_1\tilde{\eta}) = 0 \quad (35)$$

Inserting the degeneracy conditions this becomes

$$8c_2D_0^2 + 80c_1D_0 + 280c_0 = 0 \quad (36)$$

It is easy to check (once the conditions in Eq. (33) are taken into account) that this quadratic equation in D_0 has the same roots as Eq. (34) and so they are compatible.

It is worth to notice that, unlike the static vacuum Chern-Simons worm-hole of [8] in which there is a certain degree of degeneracy in the metric¹, in the present degenerate case when there is a non-vanishing torsion on the Σ_2 sub-manifold the indeterminacy is completely lifted. However, in the zero torsion sector of the theory, when conditions in Eq. (33) hold, the equations of motion would be under-determined (since Eqs. (21), (22) and (23) are identically fulfilled). This makes manifest the important role of torsion in removing degeneracies.

In order to display in a clear way the structure of the wormhole, the following change of coordinates is useful (it is worth to remember here that both D_0 and $\gamma - (\delta_{(1)})^2$ are negative in the case of a wormhole):

$$\frac{dr^2}{g^2} = \frac{dr^2}{(-D_0)r^2 - (\delta_{(1)})^2 + \gamma} = d\rho^2 \Rightarrow \quad (37)$$

$$r = r_G \cosh\left((-D_0)^{1/2} \rho\right) \quad (38)$$

where r_G will be defined in a moment (see Eq. (41) below). Thus, the wormhole metric is

$$ds^2 = -dt^2 + d\rho^2 + r_G^2 \cosh^2\left((-D_0)^{1/2} \rho\right) d\Sigma_1^2 + d\Sigma_2^2, \quad (39)$$

$$T^i = \frac{\delta_{(1)}}{r_G \cosh\left((-D_0)^{1/2} \rho\right)} \epsilon^{ijk} e_j e_k, \quad T^a = K_2 \epsilon^{abc} e_b e_c, \quad (40)$$

¹One can also see that even adding a "BPS" torsion of the form in Eq. (11) to the static vacuum Chern-Simons worm-hole the degeneracy, in general, is not removed.

where the range of the coordinate ρ extends from $-\infty$ to $+\infty$, D_0 is one of the roots of Eq. (36), the throat is located at $\rho = 0$ and the throat radius r_G is:

$$r_G = \left(\frac{(\delta_{(1)})^2 - \gamma}{(-D_0)} \right)^{1/2}. \quad (41)$$

A wormhole metric with flat $\rho-t$ plane, as the above one, in four dimensions is a vacuum solution of conformal gravity [15]. The constant of integration $\delta_{(1)}$ (which characterizes the strength of the torsion in the i directions) is not fixed by the equations of motion so that, by varying $\delta_{(1)}$, one can obtain effective five dimensional wormholes of any radius. It is worth to point out that in order to have a wormhole solution the effective curvature of the base manifold Σ_1 which is given by $\tilde{\gamma}$ must be negative. Of course, in the metric of the base manifold $d\Sigma_1^2$ (which is made out of the vielbeins which do not receive torsion corrections in this framework) it only enters the Riemannian curvature γ . This means that one can have a base manifold with positive constant Riemannian curvature γ provided a non-zero torsion concentrated on the base manifold makes $\tilde{\gamma}$ negative.

It is worth to recall here a known but important point (see [16], [17], [18]): in the case in which the Riemannian curvature of the base manifold γ is negative, in order to get a wormhole instead of Σ_1 itself, one has to consider the quotient $\hat{\Sigma}_1$ of the base manifold by a freely acting discrete subgroup Γ (otherwise there would be only one asymptotic region)

$$\hat{\Sigma}_1 = \Sigma_1/\Gamma;$$

indeed, the local expression of the gravitational field is the same as with Σ_1 .

In the case in which the base manifold has negative curvature the effective 5-D metric has locally (and hence also asymptotically on both side of the throat for $\rho \rightarrow \pm\infty$) the form $R \times H_4$. In the case in which the base manifold has positive constant curvature the effective 5-D metric is only asymptotically locally of the form $R \times H_4$. Thus, in both cases the asymptotic metric is the same on both side of the throat. The only qualitative difference in the wormhole solutions in which the metric function f is not constant is that the asymptotic is, in general, different on the two sides $\rho \rightarrow \pm\infty$.

5.2 Non-degenerate wormholes

In this non-degenerate case Eqs. (29), (30) and (31) imply that

$$g^2 = -D_0 r^2 + \gamma.$$

It is trivial to see that $B = D_0$ (B is defined in Eq. (17)) and that Eq. (23) is identically satisfied due to Eq. (29). The equation Eq. (24) is a hyper-geometric like equation for f (as it can be checked by substituting the above expression of g^2 into Eqs. (17), (18) and (24)): thus, wormhole solutions correspond to hyper-geometric functions without zeros. The simplest solution (which, nevertheless, manifests all the expected non-trivial features) is $f = \text{const}$. In this case $F = A = 0$ and Eq. (24) reduces to

$$D_0^2(12c_2 + 12c_3\tilde{\eta}) + D_0(120c_1 + 24c_2\tilde{\eta}) + 420c_0 + 20c_1\tilde{\eta} = 0 \quad (42)$$

Thus, in the zero torsion sector, one gets a wormhole provided the base manifold Σ_1 is compact and of constant negative curvature (since γ has to be negative). As it has been already explained, this can be achieved with the procedure outlined in [16], [17], [18]. Therefore, the vacuum static wormhole in the non-degenerate case is

$$ds^2 = -dt^2 + d\rho^2 + \left(\frac{-\gamma}{(-D_0)} \right) \cosh^2 \left((-D_0)^{1/2} \rho \right) d\Sigma_1^2 + d\Sigma_2^2, \quad (43)$$

where we have again used the transformation in Eq. (37).

Also in this case in which the base manifold has negative curvature the effective 5-D metric has locally (and hence also asymptotically on both side of the throat for $\rho \rightarrow \pm\infty$) the form $R \times H_4$. Thus, the asymptotic metric is the same on both side of the throat. Even in this case, the wormhole solutions in which the metric function f is not constant only differ from the ones in Eq. (43) in that the asymptotic is, in general, different on the two sides of the throat $\rho \rightarrow \pm\infty$.

The ij component of the torsion equations would imply the condition $5c_1 + 3c_2\tilde{\eta} = 0$ which is one of the degeneracy conditions in Eq. (33) and therefore such a case is excluded here so that, in this non-degenerate case, $T^i = 0$. The ab component (that is, Eq. (26)) gives

$$\epsilon_{ab} = 12c_3 D_0^2 + 24c_2 D_0 + 20c_1 = 0 \quad (44)$$

Thus, the wormhole metric reads

$$ds^2 = -dt^2 + d\rho^2 + \left(\frac{-\gamma}{(-D_0)} \right) \cosh^2 \left((-D_0)^{1/2} \rho \right) d\Sigma_1^2 + d\Sigma_2^2, \quad (45)$$

$$T^a = K_2 \epsilon^{abc} e_b e_c, \quad (46)$$

where it is worth to stress that, unlike the previous degenerate case, the throat radius is fixed by the coupling constant of the theory through D_0 (see Eq. (48) below).

It is important to notice that the Eqs. (29), (30), (42) and (44) taken together imply an extremely awkward set of constraints on the coupling constants which are impossible to solve analytically even with the program MATHEMATICA. It is more illuminating to discuss one simple case (i.e. when $c_3 = 0$) in which the constraints simplify and can be solved explicitly. This corresponds to an Einstein-Gauss-Bonnet theory. In this case inserting $c_3 = 0$ in Eqs. (29), (30), (42) and (44) one gets

$$c_0 = \frac{(25c_1^2 + 25c_1c_2\tilde{\eta} + 9c_2^2\tilde{\eta}^2)}{(35c_2)} \quad (47)$$

$$D_0 = -\frac{(20c_1)}{(24c_2)} \quad (48)$$

$$\eta = -\frac{(25c_1)}{(18c_2)} \quad (49)$$

One can also check that the expression found here for D_0 in Eq. (48) is consistent with the definition (30) and that for this choice of the coupling constants the theory has two distinct maximally symmetric eight dimensional vacua with cosmological constants $\Lambda_{1,2}$

$$\Lambda_1 = \frac{(-21c_1 - 2\sqrt{14}c_1)}{(42c_2)}; \quad \Lambda_2 = \frac{(-21c_1 + 2\sqrt{14}c_1)}{(42c_2)} \quad (50)$$

Indeed, these solutions can be seen as effective vacuum five dimensional wormholes interpolating between two asymptotic region where the spatial sections $t = \text{const}$ have the same (constant) curvature (due to the fact that both g_{tt} and g_{rr} are equal to one in Eq. (45)). In the "torsionless" case in five dimensions, vacuum wormhole solutions have been constructed in the Chern-Simons case [8]: when such wormholes have the base manifold of constant

curvature, the metric component g_{tt} is not fixed by the equations of motion (this is a typical sign of the enhanced gauge symmetry of Chern-Simons theory). However, in the present non-degenerate case, such degeneracies are completely absent since all the metric components are fixed by the equations of motion. In the previous case of the degenerate wormhole (in which the conditions in Eq. (33) are fulfilled) the degeneracies are avoided provided the torsion is non-vanishing in the Σ_2 sub-manifold.

6 How to cross the throat?

An important issue when dealing with wormholes is their "navigableness". Indeed one of the most natural questions which arises when dealing with wormholes is if a timelike or null geodesic can pass through the throat. In the case that only non-geodesic curves can cross the wormhole means that a hypothetical astronaut needs an engine to cross the wormhole. It is a known fact that torsion couples to the spin of a particle. This means that in the discussion of the "navigableness" of the wormhole one must distinguish between scalars and spinning particles. This opens the intriguing possibility that a wormhole with torsion can act as a geometric filter distinguishing between scalars and spinors and also between the helicities of particles.

We will therefore begin first to study the "navigableness" for scalar particles. In the case of nonzero torsion one has two different definitions of geodesic curve for a scalar which in general do not coincide. The first possible definition of geodesic is as the curve which extremizes the particle action

$$I = \int ds^2 \quad (51)$$

In this definition the torsion does not enter explicitly. The other possible definition of geodesic is given as an unaccelerated autoparallel curve for which the connection Γ_{jk}^k and so the torsion enter directly. This implies that the two definitions are in general not equivalent. In the latter case the geodesic equation is given by

$$\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda = 0 \quad (52)$$

where the connection symbols can be decomposed as

$$\Gamma_{\nu\lambda}^\mu = \hat{\Gamma}_{\nu\lambda}^\mu + K_{\nu\lambda}^\mu \quad (53)$$

in which $\hat{\Gamma}_{\nu\lambda}^\mu$ are the Christoffel symbols and $K_{\nu\lambda}^\mu$ is the contorsion tensor. In our ansatz for the torsion in Eq. (11) the contorsion takes the simple form

$$K_{\nu\lambda}^\mu = e_i^\mu e_\nu^j K_{j\lambda}^i \approx -e_i^\mu e_\nu^j \epsilon_{jk}^i e_\mu^k \approx -\epsilon_{\nu\lambda}^\mu. \quad (54)$$

This implies that the correction to the autoparallel geodesic equation is identically zero due to the contraction with $\dot{x}^\nu \dot{x}^\lambda$. One concludes that for this special form of the contorsion the two definitions of scalar geodesics coincide.

Naively one should expect that scalar particles should not feel the torsion. In fact, torsion enters directly the metric through the constant $\delta_{(1)}$ modifying the size of the throat. Along a geodesic one can normalize the tangent vector as follows

$$g_{AB} (\partial_\tau X^A) (\partial_\tau X^A) = -k$$

where k is zero or one for a lightlike and a timelike particle respectively and we will assume that the tangent vector has no component along the extra dimensions a (so that $A = 1, 0, i$). For the sake of simplicity, let us consider a geodesic in which only one angular coordinate (say, ϕ) is not constant: such a case is enough to show the effects of torsion on the particles dynamics. The effective equation reads

$$-1 = -\dot{t}^2 + \dot{\rho}^2 + (r_G)^2 \cosh^2((-D_0)^{1/2} \rho) \dot{\phi}^2 \quad (55)$$

where r_G is defined in Eq. (41). There are two Killing vectors $\xi_1 = \partial_t$ and $\xi_2 = \partial_\phi$ so there are two conserved quantities along the geodesic worldline u_μ which are $E = \xi_1 \cdot u$ and $J = \xi_2 \cdot u$. The radial motion reduces to a one dimensional problem in an effective potential:

$$\dot{\rho}^2 + V_{eff} = E^2; \quad V_{eff} = 1 + \frac{J^2}{(r_G)^2 \cosh^2((-D_0)^{1/2} \rho)}. \quad (56)$$

It is worth to note that particles with a vanishing angular momentum J do not feel any potential barrier (the non-trivial term of V_{eff} is purely centrifugal in nature²). When E is large enough,

$$E^2 > 1 + \frac{J^2}{(r_G)^2},$$

²On the other hand, wormhole solutions in which f is not constant have, in general, a non-trivial potential barrier even for purely radial motion: see, for instance, [8].

timelike geodesic with $J \neq 0$ can cross the wormhole's throat. The interesting feature which discloses the physical effects of torsion on scalar particles is that when $(r_G)^2$ becomes larger and larger (so that the strength of the torsion becomes very large as well) the centrifugal barrier correspondingly becomes lower and lower: it is then "easier" for an effective five dimensional geodesic to cross the throat. It is also interesting to note that the opposite limit is the one in which $(r_G)^2$ is small: in this case, the base manifold becomes almost teleparallelized³ and the barrier becomes very high: this "almost" prevents scalar particles with a $J \neq 0$ from crossing the throat⁴.

Let us now see what happens in the case of a spinning particle which has no components in the extra dimensions. In principle, one should study the corresponding (classical or quantum) equations of motion to determine the effects of torsion on the dynamics of spin. However, there is a quite general qualitative argument which provides one with a clear intuitive picture of the dynamical effects of a purely axial torsion. It is well known (see, for a detailed review, [14]) that, in four dimensions, the classical equation of motion of a spinning particle with intrinsic angular momentum \vec{J} in a background with torsion represented by the axial vector \vec{S} (we will neglect here the effects of curvature to isolate the torsion contribution) can be written schematically

$$\partial_t \vec{J} \approx \chi (\vec{J} \times \vec{S}),$$

(where χ is some effective coupling constant) which corresponds to the following interaction Hamiltonian

$$H_{st} \approx -\chi \vec{J} \cdot \vec{S}. \quad (57)$$

Thus, on very general grounds, one can say that if the torsion effective coupling constant χ is very large compared to other scales⁵ \vec{J} "is polarized" by \vec{S} : states in which \vec{J} is parallel to \vec{S} have energies much lower than the others states.

³Since the total connection $(\omega_{tot})^{ij}$ almost vanishes being proportional to $\tilde{\gamma}$ and $\tilde{\gamma} \simeq 0$, see Eq. (13).

⁴A smaller and smaller $|\tilde{\gamma}|$ would correspond to make the neck narrower and narrower: the limiting case in which $|\tilde{\gamma}|$ vanishes (in which case the amplitude of the neck at the throat vanishes as well) has been called "space-time horn" in [8] and, strictly speaking, does not correspond to a wormhole.

⁵In the present case one may hope to achieve this condition since the constant of integration $\delta_{(1)}$ is not constrained by the field equations.

Even if the above argument holds in four dimensions, it is known that also in higher dimensions the coupling of the torsion with spin is similar. Let us discuss, for instance, the case of the Maxwell Lagrangian in a background with torsion:

$$\begin{aligned} L &= \sqrt{-g} d^D x \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} = L_0 + L_I, \\ L_I &= \sqrt{-g} d^D x (2F \cdot A \cdot T + A \cdot A \cdot T \cdot T), \\ \hat{F}_{\mu\nu} &= 2\nabla_{[\mu} A_{\nu]} = F_{\mu\nu} - T_{\mu\nu}^\rho A_\rho \end{aligned}$$

where D is the number of spacetime dimensions, A_ν is the gauge field, $F_{\mu\nu}$ is the torsion free field strength, L_0 is the torsion free Maxwell Lagrangian and L_I is the spin-torsion interaction term⁶ and obvious contractions have been understood in L_I . In the limit in which the torsion is large the energy of the spin-torsion interaction is

$$H_I \approx -\sqrt{-g} d^D x (A \cdot A \cdot T \cdot T).$$

One can "minimize" the above interaction energy by choosing the polarization of A_μ which maximizes the (integral of the) product $A \cdot A \cdot T \cdot T$. These kinds of effects appear whenever the Levi-Civita connection acting on tensor fields is corrected by the torsion.

Thus, the throat can really act as a geometrical filter which distinguishes scalars and the different polarization states of spinning particles: such "polarization" effect is maximum at the throat since there the effective strength χ_{eff} of the torsion (see Eq. (40))

$$\chi_{eff} \approx \frac{\delta_{(1)}}{r_G \cosh \left((-D_0)^{1/2} \rho \right)}$$

has a maximum while being very small when $|\rho|$ is large.

7 Black Holes

Let us shortly describe Black hole solutions which correspond to the case $f = g$ in which Eqs. (21) and (22) becomes identical. The following simpli-

⁶Indeed, the above Lagrangian is not satisfactory as a fundamental Lagrangian for the electromagnetic field since it explicitly breaks gauge invariance when torsion does not vanish. However, our scope here is only to show how torsion naturally acts as a "polarizator" on particles with spin. Indeed for our example one could also have chosen a massive spin one field obtaining the same conclusions as in the massless case

fications in the curvature two forms occurs

$$F = \frac{1}{2}(-f^2)'' = \frac{1}{2}Z'' \quad ; \quad B = \frac{1}{2r}(-f^2)' = \frac{1}{2r}(Z)' \quad ; \quad A = -B \quad (58)$$

where we have introduced the function Z defines as

$$Z \equiv \tilde{\gamma} - f^2 = Dr^2 \quad (59)$$

Eqs. (21) and (22) now read

$$\begin{aligned} \frac{1}{2r}Z' \left[20c_1 + 12c_2\tilde{\eta} + \frac{Z}{r^2}(4c_2 + 12c_3\tilde{\eta}) \right] + 140c_0 \\ + 20c_1\tilde{\eta} + \frac{Z}{r^2}(20c_1 + 12c_2\tilde{\eta}) = 0. \end{aligned} \quad (60)$$

7.1 The generic case

In the generic case, one express $\frac{1}{2r}Z'$ as rational function of $\frac{Z}{r^2}$:

$$-\frac{Z'}{2r} = \frac{140c_0 + 20c_1\tilde{\eta} + \frac{Z}{r^2}(20c_1 + 12c_2\tilde{\eta})}{\left[20c_1 + 12c_2\tilde{\eta} + \frac{Z}{r^2}(4c_2 + 12c_3\tilde{\eta}) \right]}. \quad (61)$$

The other components of the equations of motion are

$$\begin{aligned} \frac{Z''}{2} \left[20c_1 + 12c_2\tilde{\eta} + \frac{Z}{r^2}(4c_2 + 12c_3\tilde{\eta}) \right] + \frac{Z'}{r} [40c_1 + 24c_2\tilde{\eta}] \\ + \left(\frac{Z'}{2r} \right)^2 [8c_2 + 24c_3\tilde{\eta}] + \frac{Z}{r^2} [20c_1 + 12c_2\tilde{\eta}] + 420c_0 + 60c_1\tilde{\eta} = \epsilon_i = 0 \end{aligned} \quad (62)$$

$$\begin{aligned} \frac{Z''}{2} \left[20c_1 + 4c_2\tilde{\eta} + 12\frac{Z}{r^2}(c_2 + c_3\tilde{\eta}) \right] + \frac{Z'}{r} \left[60c_1 + 12c_2\tilde{\eta} + 12\frac{Z}{r^2}(c_2 + c_3\tilde{\eta}) \right] \\ + 24 \left(\frac{Z'}{2r} \right)^2 [c_2 + c_3\tilde{\eta}] + \frac{Z}{r^2} [60c_1 + 12c_2\tilde{\eta}] + 420c_0 + 20c_1\tilde{\eta} = \epsilon_a = 0 \end{aligned} \quad (63)$$

Taking into account Eq. (61), Eq. (62) allows to express $\frac{Z''}{2}$ as a rational function $\frac{Z}{r^2}$: let us call such a function $Y_{(i)}$ to stress that it comes from the i -th component of the equations of motion

$$\epsilon_i = 0 \Rightarrow \frac{Z''}{2} = Y_{(i)} \left(\frac{Z}{r^2}, c_1, c_2, c_3, c_0 \right). \quad (64)$$

On the other hand, using Eqs. (61) and (63) one can find a different expression for $\frac{Z''}{2}$ as a rational function of $\frac{Z}{r^2}$: let us call such a function $Y_{(a)}$ to stress that it comes from the a -th component of the equations of motion

$$\epsilon_a = 0 \Rightarrow \frac{Z''}{2} = Y_{(a)} \left(\frac{Z}{r^2}, c_1, c_2, c_3, c_0 \right). \quad (65)$$

In the generic case, the two expressions⁷ for $Y_{(i)}$ and $Y_{(a)}$ are different so that one has to impose the consistency condition

$$Y_{(i)} = Y_{(a)}. \quad (66)$$

The above equation can be written as a polynomial of finite degree N in $\frac{Z}{r^2}$ whose constant coefficients are related to the coupling constants c_i and to $\tilde{\eta}$:

$$Y_{(i)} = Y_{(a)} \Rightarrow \sum_{n=0}^N a_n \left(\frac{Z}{r^2} \right)^n = 0. \quad (67)$$

This actually implies that $\frac{Z}{r^2}$ (being one of the root of a polynomial with constant coefficients) is constant:

$$Z = \alpha r^2 \Rightarrow$$

$$\begin{aligned} f^2 &= -\alpha r^2 + \tilde{\gamma}, \\ \alpha &< 0, \quad \tilde{\gamma} < 0, \end{aligned} \quad (68)$$

where α is related to the c_i and to $\tilde{\eta}$. This looks like a Chern-Simons black hole provided both α and $\tilde{\gamma}$ are negative. However, as it will be shown in a moment, unlike the Chern-Simons case [7] torsion appears directly in the metric which takes the form

$$ds^2 = -(-\alpha r^2 + \tilde{\gamma})dt^2 + \frac{dr^2}{(-\alpha r^2 + \tilde{\gamma})} + r^2 d\Sigma_1^2 + d\Sigma_2^2. \quad (69)$$

The torsion equations in this case give

$$\epsilon_{ij} = 0 \Rightarrow \alpha(4c_2 + 12c_3\tilde{\eta}) + 20c_1 + 12c_2\tilde{\eta} = 0 \quad (70)$$

⁷The explicit expressions for $Y_{(i)}$ and $Y_{(a)}$ are quite long and not very illuminating but we will not need them in the following.

$$\epsilon_{ab} = 0 \Rightarrow \alpha(4c_2 + 12c_3\alpha) - 24c_3\alpha^2 + 20c_1 + 12c_2\alpha = 0; \quad (71)$$

such equations put two further constraints on the coefficients c_i . One can see that having nonzero torsion in both the ij components as well in the ab components (that is, $\delta_{(1)} \neq 0$ and $K_2 \neq 0$ in Eqs. (15) and (16)) is not consistent since it would lead to $c_3 = c_2 = 0$.

In the case in which torsion is nonzero only in the ij components (that is, $\delta_{(1)} \neq 0$ in Eq. (15)), one can use the previous equations to express c_0 , c_1 , c_2 in function of $\tilde{\eta}$, α and c_3

$$c_0 = \frac{3\alpha^2 c_3 \tilde{\eta} (5\alpha^2 - 5\alpha\tilde{\eta} - 4\tilde{\eta}^2)}{35(4\alpha^2 - 9\alpha\tilde{\eta} + 3\tilde{\eta}^2)} \quad (72)$$

$$c_1 = \frac{3\alpha^2 c_3 \tilde{\eta} (-5\alpha + 7\tilde{\eta})}{5(4\alpha^2 - 9\alpha\tilde{\eta} + 3\tilde{\eta}^2)} \quad (73)$$

$$c_2 = \frac{3\alpha c_3 (\alpha - \tilde{\eta}) \tilde{\eta}}{4\alpha^2 - 9\alpha\tilde{\eta} + 3\tilde{\eta}^2} \quad (74)$$

From the above expressions for c_0 , c_1 , c_2 , one can see α depends on $\tilde{\eta}$: in other words, the lapse function of the effective five-dimensional black hole depends on the torsion in the extra-dimensions. It is also worth to stress that $\tilde{\gamma}$ is left undetermined by the equations of motion. The ratios $\frac{c_2^2}{c_1 c_3}$ and $\frac{c_1^2}{c_0 c_2}$ are rational functions of $\tilde{\eta}$ and $\tilde{\gamma}$ so that these solutions do not belong to the Born-Infeld class. In the case that one has a nonzero torsion only in the ab components one obtains

$$c_0 = \frac{3}{35} \alpha^2 c_3 (5\alpha - 4\tilde{\eta}) \quad (75)$$

$$c_1 = -\frac{9\alpha^2 c_3}{5} \quad (76)$$

$$c_2 = 3\alpha c_3 \quad (77)$$

7.2 Degenerate black holes

Black hole solutions in the case in which the degeneracy conditions in Eq. (32) hold are also interesting. In such a case, Eqs. (60), (62) are identically

satisfied and one is only left with Eq. (63):

$$\begin{aligned} \frac{Z''}{2} \left[20c_1 + 4c_2\tilde{\eta} + 12\frac{Z}{r^2}(c_2 + c_3\tilde{\eta}) \right] + \frac{Z'}{r} \left[60c_1 + 12c_2\tilde{\eta} + 12\frac{Z}{r^2}(c_2 + c_3\tilde{\eta}) \right] \\ + 24 \left(\frac{Z'}{2r} \right)^2 [c_2 + c_3\tilde{\eta}] + \frac{Z}{r^2} [60c_1 + 12c_2\tilde{\eta}] + 420c_0 + 20c_1\tilde{\eta} = 0 \end{aligned} \quad (78)$$

Indeed, this is a quite non-trivial non-linear differential equation which is difficult to solve in general. Anyway it is easy to show that effective five dimensional Chern-Simons black holes like in Eq. (69) can solve Eq. (78). If one searches for Chern-Simons black holes characterized by Eq. (68), a nonzero torsion in the ab components (that is, $K_2 \neq 0$ in Eq. (16)) would be inconsistent since it would lead to $c_3 = c_2 = 0$. In the case in which torsion is non-vanishing only in the ij components (that is, $\delta_{(1)} \neq 0$ in Eq. (15) while $K_2 = 0$ so that $\tilde{\eta} = \eta$) one gets

$$\alpha = \frac{-5c_2 \pm \sqrt{10}|c_2|}{15c_3} \quad (79)$$

$$c_2 = -3c_3\eta. \quad (80)$$

It is interesting to note that, in the case in which c_2 is positive Eq. (79) implies that c_3 has to be positive in order to have a black hole (since α has to be negative) and this would imply that η (which is the constant curvature of the extra-dimensions) is negative. Thus, in order to have compact extra-dimensions one has to compactify the manifold Σ_2 (the standard procedure is to quotient the hyperbolic three-dimensional constant curvature space by a freely acting discrete group Γ).

It remains the open question of finding more general black hole solutions in the degenerate case. The most natural ansatz representing further black hole solutions would be a compactified Boulware-Deser black hole [19] or a compactified Schwarzschild-(Anti)de-Sitter black hole. However one can check that those ansatz do not satisfy the above equation of motion. Moreover even modifying the exponents of the radial coordinate in the lapse function does not improve the situation. Therefore possible further black hole solutions will have a structure quite different from the one we found. Finding such solutions seems to be a highly nontrivial but interesting task and will be object of future investigation.

8 Generalized Bertotti-Robinson solutions

For the sake of completeness, here we will shortly describe a class of generalized Bertotti-Robinson spacetimes. In this case we search for solutions of the form $(A) dS_2 \times S_3 \times S_3$ so that the metric reads

$$ds^2 = \frac{l^2}{r^2}(-dt^2 + dr^2) + d\Sigma_1^2 + d\Sigma_2^2 \quad (81)$$

In this ansatz the curvature two forms are

$$R^{0a} = R^{1a} = R^{ia} = R^{0i} = R^{1i} = 0 \quad (82)$$

$$R^{01} = -\frac{1}{l^2} e^0 e^1 \quad (83)$$

$$R^{ij} = \tilde{\gamma} e^i e^j \equiv D_0 e^i e^j \quad (84)$$

$$R^{ab} = \tilde{\eta} e^a e^b \quad (85)$$

and the torsion is now

$$T^i = \delta_{(1)} \epsilon^{ijk} e_j e_k, \quad T^a = K_2 \epsilon^{abc} e_b e_c, \quad (86)$$

$$\tilde{\gamma} = \gamma - (\delta_{(1)})^2, \quad \tilde{\eta} = \eta - K_2^2,$$

where, as in the previous sections, $\delta_{(1)}$ and K_2 are constants. The equations of motion become

$$\epsilon_0 = \epsilon_1 = 140c_0 + 20c_1\tilde{\gamma} + 20c_1\tilde{\eta} + 12c_2\tilde{\gamma}\tilde{\eta} = 0$$

$$\epsilon_i = -\frac{1}{l^2} (20c_1 + 4c_2\tilde{\gamma} + 12c_2\tilde{\eta} + 12c_3\tilde{\gamma}\tilde{\eta}) + 420c_0 + 20c_1\tilde{\gamma} + 60c_1\tilde{\eta} + 12c_2\tilde{\gamma}\tilde{\eta} = 0$$

$$\epsilon_a = -\frac{1}{l^2} (20c_1 + 4c_2\tilde{\eta} + 12c_2\tilde{\gamma} + 12c_3\tilde{\gamma}\tilde{\eta}) + 420c_0 + 20c_1\tilde{\eta} + 60c_1\tilde{\gamma} + 12c_2\tilde{\gamma}\tilde{\eta} = 0$$

The torsion equations read

$$\epsilon_{ij} = \delta_{(1)} \left[-\frac{1}{l^2} (4c_2 + 12c_3\tilde{\eta}) + 20c_1 + 12c_2\tilde{\eta} \right] = 0$$

$$\epsilon_{ab} = K_2 \left[-\frac{1}{l^2} (4c_2 + 12c_3\tilde{\gamma}) + 20c_1 + 12c_2\tilde{\gamma} \right] = 0$$

When both $\delta_{(1)}$ and K_2 are non-vanishing, the two torsion equations imply that or $\tilde{\gamma} = \tilde{\eta}$. If one is interested in the cases in which $\tilde{\gamma} \neq \tilde{\eta}$ then one of the two torsions is zero (i.e. or $\delta_{(1)} = 0$ or $K_2 = 0$)

Let us suppose for the sake of simplicity that only $\delta_{(1)}$ is non-vanishing so that $\tilde{\eta} = \eta$. The equations of motion allow to explicitly specify the couplings c_0, c_1, c_2, c_3 in terms of l , $\tilde{\eta}$ and $\tilde{\gamma}$ (so that only three of the four c_i are independent⁸):

$$c_3 = -\frac{(\tilde{\gamma}^2(1 + 3\tilde{\gamma}l^2 + 6\eta(-\eta + \tilde{\gamma})l^4))}{(-2\eta + \tilde{\gamma} + 3(-2\eta^2 + \tilde{\gamma}^2)l^2 + 12\eta\tilde{\gamma}(-\eta + \tilde{\gamma})l^4)} \quad (87)$$

$$c_1 = \frac{3\eta\tilde{\gamma}^2(\eta + 2\tilde{\gamma} + 9\eta\tilde{\gamma}l^2)}{5(2\eta - \tilde{\gamma} + 3(2\eta^2 - \tilde{\gamma}^2)l^2 + 12\eta(\eta - \tilde{\gamma})\tilde{\gamma}l^4)} \quad (88)$$

$$c_2 = -\frac{3\eta\tilde{\gamma}^2(1 + (2\eta + \tilde{\gamma})l^2)}{2\eta - \tilde{\gamma} + 3(2\eta^2 - \tilde{\gamma}^2)l^2 + 12\eta(\eta - \tilde{\gamma})\tilde{\gamma}l^4} \quad (89)$$

$$c_0 = -\frac{(3\eta\tilde{\gamma}^2(\eta^2 + 2\tilde{\gamma}^2 + 3\eta\tilde{\gamma}(\eta + 2\tilde{\gamma})l^2))}{(35(2\eta - \tilde{\gamma} + 3(2\eta^2 - \tilde{\gamma}^2)l^2 + 12\eta(\eta - \tilde{\gamma})\tilde{\gamma}l^4))}. \quad (90)$$

It is worth to point out that the fractions $\frac{c_2^2}{c_1 c_3}$ and $\frac{c_1^2}{c_0 c_2}$ are rational functions of η and $\tilde{\gamma}$ so that, in general, the above solutions do not belong to Born-Infeld theory.

In the case in which both torsions are switched on (namely $\delta_{(1)}$ and K_2 non-vanishing) so that $\tilde{\gamma} = \tilde{\eta}$ the above equations remain valid.

A quite non-trivial characteristic of the present construction is that the sizes as well as the curvatures of the three factors (A) dS_2 , S_3 , and the second factor S_3 can have in principle very different values as it is clear from Eqs. (87), (88), (89) and (90). Namely, nothing prevents, for instance, η from being of the same order of $\tilde{\gamma}$ and, at the same time, the AdS radius l^2 from being much larger than the size of the extra dimensions. The remarkable fact is that this can be achieved *in vacuum*: the only price to pay is that c_3 turns out to be much larger than the other c_i in this limit (since only in the expression for c_3 in Eq. (87) the highest power in l in the numerator is the same as in the denominator).

⁸The expressions below have been checked with the software MATEMATICA.

9 Conclusions

In this paper we constructed exact vacuum solutions with torsion in 8-D Lovelock theory of the form $M_5 \times S_3$, which can then be seen as effective five dimensional geometries. The solutions that have been found are static vacuum effective five-dimensional wormholes, black holes and generalized Bertotti-Robinson solutions in which the three compact extra-dimensions play a "spectator" role. All these solutions can carry nontrivial torsion even in the non-Born-Infeld case. The wormhole "navigableness" has been discussed and it has been shown that the torsion has very different effects on scalar or spinning particles. A huge amount of torsion improves the "navigableness" for scalars while acting as a "polarizator" on spinning particles.

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References

- [1] Lovelock D., *J. Math. Phys.* **12** (1971) 498.
- [2] Garraffo C. and Giribet G., *The Lovelock Black Holes*, arXiv:0805.3575.
- [3] R. Troncoso, J. Zanelli, *Gauge supergravities for all odd dimensions*, *Int.J.Theor.Phys.* **38** (1999) 1181; *Higher dimensional gravity, propagating torsion and AdS gauge invariance*, *Class.Quant.Grav.* **17** (2000) 4451.

- [4] Zanelli J., *Lecture notes on Chern-Simons (super-)gravities*, hep-th/0502193.
- [5] F. Canfora,, “Some solutions with torsion in Chern-Simons gravity and observable effects,” arXiv:0706.3538 [gr-qc].
- [6] Canfora F., Giacomini A., Willison S., ” *Some exact solutions with torsion in 5-D Einstein-Gauss-Bonnet gravity*” *Phys.Rev.* **D76**: 044021, 2007.
- [7] Canfora F., Giacomini A., Troncoso R., ” *Black holes, parallelizable horizons and half-BPS states for the Einstein-Gauss-Bonnet theory in five dimensions*” *Phys.Rev.* **D77**: 024002, 2008.
- [8] G. Dotti, J. Oliva, R. Troncoso, *Phys.Rev.* **D75** (2007) 024002.
- [9] G. Dotti, J. Oliva, R. Troncoso, *Phys.Rev.* **D76**(2007) 064038.
- [10] G. Dotti, J. Oliva, R. Troncoso ”Wormholes and black holes with non-trivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum” CECS-PHY-07/21.
- [11] M. Visser, *Lorentzian Wormholes*, Springer (1996), New York.
- [12] H. Maeda and M. Nozawa, *Phys. Rev.* **D78**, 024005 (2008).
- [13] H. Lu, J. Mei, *Ricci flat and charged wormholes in five dimensions*, arXiv:0806.3111.
- [14] I. L. Shapiro, *Physics Report* **357** (2002), 113.
- [15] J. Oliva, D. Tempo, R. Troncoso ”Static wormholes in vacuum and electrically charged without charge” CECS-PHY-08/09.
- [16] Aros R., Martinez C., Troncoso R., and Zanelli J., ” *Supersymmetry of gravitational ground states*”, *J. High Energy Phys.* **05** (2002) 020.
- [17] J. Maldacena, L. Maoz, *J. High Energy Phys.* **0402** (2004) 008.
- [18] D. H. Correa, J. Oliva, R. Troncoso, *Stability of asymptotically AdS wormholes in vacuum against scalar field perturbations* arXiv:0805.1513.
- [19] D. G. Boulware and S. Deser, *Phys. Rev. Lett.* **55**, 2656 (1985).